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[1] Whitfield, J. D. and Wolny, W., "Hypersonic static stability of blunt, slender cones," Arnold Eng. Dev. Center TDR-62-166 (August 1962).

[2] Henderson, A., Jr., "Recent investigations of the aerodynamic characteristics of general and specific lifting and non-lifting configurations at Mach 24 in helium, including air-helium simulation studies," *High Temperature Aspects of Hypersonic Flow* (Pergamon Press, Oxford, to be published).

Radial Vibrations of Thick-Walled Orthotropic Cylinders

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In the design of modern missiles and space vehicles, increasing use is being made of the newer materials, such as reinforced plastics, whisker materials, and fiber-reinforced metals. These materials are essentially elastically orthotropic; that is, the Young's modulus of elasticity and Poisson's ratio differ in the three mutually perpendicular directions. Approximate formulas are developed here for the natural wavelengths associated with free radial vibrations of a thick-walled, infinitely long, orthotropic cylinder. The method has been employed previously by McFadden[1] for the case of isotropic cylinders.

For purely radial vibrations, the particle displacement $u(r,t)$ is governed by the equation

$$c_{11}(\partial^2 u/\partial r^2) + (c_{11}/r)(\partial u/\partial r) - c_{22}(u/r^2) = \rho(\partial^2 u/\partial t^2) \quad (1)$$

where r is the radial coordinate, t the time, and

$$c_{11} = \eta E_r(1 - \nu_{\theta z}\nu_{z\theta})$$
$$c_{22} = \eta E_\theta(1 - \nu_{zr}\nu_{rz})$$
$$1/\eta = 1 - \nu_{\theta r}\nu_{r\theta} - \nu_{rz}\nu_{zr} - \nu_{z\theta}\nu_{\theta z} - \nu_{rz}\nu_{z\theta}\nu_{\theta r} - \nu_{zr}\nu_{r\theta}\nu_{\theta z} \quad (2)$$

Note that, as opposed to the two elastic constants E and ν for an isotropic material, nine elastic constants E_r, E_θ, $\nu_{r\theta}$, etc., are required to describe the behavior of an orthotropic material.

If one assumes $u(r,t)$ in the form

$$u(r,t) = U(r)e^{i\omega t} \quad (3)$$

then $U(r)$ satisfies the equation

$$r^2(d^2U/dr^2) + r(dU/dr) + (k^2r^2 - n^2)U = 0 \quad (4)$$

where

$k = \omega/C_c$ is a wave number
ω = angular frequency
$C_c = (c_{11}/\rho)^{1/2}$ is the phase velocity of compressional waves
$n^2 = c_{22}/c_{11}$

Equation (4) is the familiar Bessel's equation of order n and argument kr. The general solution is

$$U(r) = AJ_n(kr) + BY_n(kr) \quad (5)$$

where A and B are constants of integration.

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[1] McFadden, J. A., "Radial vibrations of thick walled hollow cylinders," J. Acoust. Soc. Am. 26 (September 1954).

For free motions, the radial stress σ_{rr} vanishes on the inner and outer surfaces $r = a,b$, respectively. Hence the boundary conditions in terms of the displacement are

$$c_{11}(dU/dr) + c_{12}(U/r) = 0 \text{ on } r = a,b \quad (6)$$

where

$$c_{12} = \eta E_\theta(\nu_{r\theta} + \nu_{rz}\nu_{z\theta})$$

These boundary conditions result in the frequency equation given by

$$\begin{vmatrix} ka\,J_{n-1}(ka) - \beta\,J_n(ka) & ka\,Y_{n-1}(ka) - \beta\,Y_n(ka) \\ kb\,J_{n-1}(kb) - \beta\,J_n(kb) & kb\,Y_{n-1}(kb) - \beta\,Y_n(kb) \end{vmatrix} = 0 \quad (7)$$

with $\beta = n - c_{12}/c_{11}$.

Solution of Frequency Equation for Extensional Mode

Consider first the extensional mode. The frequency equation may be written as

$$F(ka) = F(kb) \quad (8)$$

where

$$F(x) = \frac{x\,J_{n-1}(x) - \beta\,J_n(x)}{x\,Y_{n-1}(x) - \beta\,Y_n(x)} \quad (9)$$

This function $F(x)$ is zero at $x = 0$ and increases with x until a maximum is reached at $x = x_0$ given by

$$x_0 = [n^2 - (n - \beta)^2]^{1/2} \quad (10)$$

From here, the function decreases with x approaching $-\infty$ and then begins at $+\infty$ and decreases, etc. The function $F(x)$ may be developed now in a Taylor series about the point x_0 to yield the following expansion:

$$F(x) = F(x_0) + \frac{(x - x_0)}{1!}F'(x_0) + \frac{(x - x_0)^2}{2!}F''(x_0) + \ldots \quad (11)$$

If one employs this expansion in the frequency equation [Eq. (8)] and writes

$$b = a(1 + \delta) \qquad ka = x_0 \sum_{s=0}^{\infty} a_s\,\delta^s \quad (12)$$

the constants a_s are obtained by solving successively equations of higher order in δ, and the resulting wavelength $\lambda = 2\pi/k$ is

$$\lambda = (2\pi a/x_0)[1 + (\delta/2) + (\tfrac{1}{8})(m - 1)\delta^2 - (\tfrac{1}{16})(m - 1)\delta^3 + 0(\delta^4)] \quad (13)$$

where

$$m = \tfrac{2}{3} - (\tfrac{4}{3})(n - \beta)$$

The approximate expression (13) for λ may be identified with the mth-order mean radius and written as

$$\lambda \simeq (2\pi/x_0)[(a^m + b^m)/2]^{1/m} + 0(\delta^4) \quad (14)$$

Note that for an isotropic material

$$n = 1 \qquad \beta = (1 - 2\nu)/(1 - \nu) \qquad m = \frac{2(1 - 3\nu)}{3(1 - \nu)}$$

and the result reduces to that developed by McFadden.

Thickness Modes

Consider now the thickness modes of a hollow orthotropic cylinder. The Bessel functions in Eq. (7) may be replaced by their asymptotic expansions with the result

$$\tan k(b - a) \sim (b - a)[8\beta + 4(n - 1)^2 - 1]/(8kab) \quad (15)$$

Writing $h = b - a$, an approximate solution to this equation is

$$kh \sim q\pi + (h^2/8q\pi ab)[8\beta + 4(n-1)^2 - 1] \qquad q = 1,2,3, \ldots \quad (16)$$

For an infinite plate of thickness h, $a,b \to \infty$ and $kh = q\pi$, $q = 1,2,3\ldots$, which would be the plane-wave solution for an orthotropic plate.

The forementioned approximate formulas (14) and (16) have been checked by comparisons with the exact solutions, and the results are found to be accurate within 5% for values of δ up to $\frac{1}{2}$.

An Error Analysis in the Digital Computation of the Autocorrelation Function

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Introduction

Let $f_1(t)$ be a function of time which is defined and continuous and satisfies all hypotheses of the ergodic theorem[4] on $-\infty < t < +\infty$. Choose a finite record, say $-T \leqslant t \leqslant +T$, and let $f_1(t)$ satisfy the quasi-ergodic hypothesis[3] on $-T$ to $+T$. The purpose of this paper is to deduce a relationship between error ϵ and maximum time lag τ_m in the digital computation of the autocorrelation function of $f_1(t)$ over $-T \leqslant t \leqslant T$. It thus will be demonstrated that the maximum time lag τ_m should not exceed 5 to 10% of the total time record, as suggested by Blackman and Tukey.[1] Details of the analysis that follows can be found in Ref. 2.

Analysis of the Problem

The unnormalized autocorrelation function of $f_1(t)$ can be defined by the equation

$$A_{11}(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_1(t) f_1(t+\tau)\,dt \qquad (1)$$

where τ is the time lag.

If only a finite record is available, then $A_{11}(\tau)$ is approximated by

$$A_{11}(\tau) \doteq \frac{1}{2T}\int_{-T}^{T} f_1(t) f_1(t+\tau)\,dt \qquad (2)$$

A typical graph of $f_1(t)$ and $f_1(t+\tau)$ is shown in Fig. 1.

Define the integrand of Eq. (2) as

$$F(t) = f_1(t) f_1(t+\tau) \qquad (3)$$

A typical graph of $F(t)$ is shown in Fig. 2.

To approximate digitally $A_{11}(\tau)$ from Eq. (2), divide the interval $-T \leqslant t \leqslant T - \tau$ into $K - m$ equispaced intervals, where m is a positive integer such that

$$\tau = m(2T/K) \qquad (4)$$

Also note that each of the equispaced subintervals in Fig. 2 is of length

$$\Delta t = 2T/(K-m) \qquad 0 \leqslant m < K \quad (5)$$

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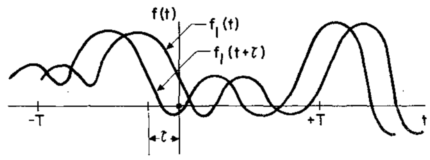

Fig. 1 Graph of $f_1(t)$ and $f_1(t+\tau)$

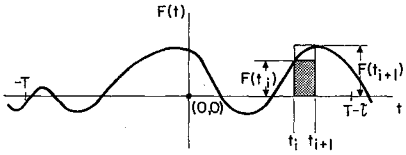

Fig. 2 Graph of $f_1(t)\,f_1(t+\tau)$

Without loss of generality, define

$$m = [K\theta] \qquad 0 \leqslant \theta < 1 \quad (6)$$

where $[K\theta]$ means the greatest integer in $K\,\theta$.

The area A_i under $F(t)$ over $t_i \leqslant t \leqslant t_{i+1}$ can be described by the inequality

$$F(t_i)\,\Delta t \leqslant A_i \leqslant F(t_{i+1})\,\Delta t \qquad (7)$$

Rearranging inequality (7) slightly and summing over $-T \leqslant t \leqslant T - \tau$ gives

$$0 \leqslant \sum_{i=0}^{K-m-1} [A_i - F(t_i)\,\Delta t] \leqslant \sum_{i=0}^{K-m-1} [F(t_{i+1}) - F(t_i)]\,\Delta t \qquad (8)$$

In inequality (8), let

$$S = \sum_{i=0}^{K-m-1} [F(t_{i+1}) - F(t_i)]$$
$$= [F(t_1) - F(t_0)] + [F(t_2) - F(t_1)] + \ldots + [F(t_{K-m-1}) - F(t_{K-m-2})] + [F(t_{K-m}) - F(t_{K-m-1})]$$

Then

$$S = F(t_{K-m}) - F(t_0) \qquad (9)$$

Substitute Eqs. (5) and (9) into inequality (8), and then introduce the factor $1/2T$ to get the following inequality:

$$0 \leqslant \frac{1}{2T}\sum_{i=0}^{K-m-1} [A_i - F(t_i)\,\Delta t] \leqslant [F(t_{K-m}) - F(-T)]\left(\frac{1}{K-m}\right) \qquad (10)$$

where $F(t_0) = F(-T)$ from Fig. 2.

Application of the fundamental theorem of integral calculus to Eq. (10) will yield an integral expression for $A_{11}(\tau)$ on $-T \leqslant t \leqslant T - \tau$. A measure of the error in the approximation of $A_{11}(\tau)$ is given by inequality (10). Thus

$$\epsilon = |[F(t_{K-m}) - F(-T)][1/(K-m)]| \qquad (11)$$

Let c be the maximum value of $|[F(t_{K-m}) - F(-T)]|$ for all $0 \leqslant m < K$, and then

$$\epsilon \leqslant c/(K-m) \qquad (12)$$

From Eq. (4)

$$\epsilon(\tau) \leqslant c/[K(1 - \tau/2T)] \qquad (13)$$

Inequality (13) defines an error region as shown in Fig. 3.